International Journal of Theoretical Physics, Vol. 20, No. 6, 1981

Geometrical Derivation of the Klein–Gordon Equation

Daniel C. Galehouse[1]

Machine Shop, Department of Physics, University of California, Berkeley, California

Received March 13, 1980

In order to integrate quantum mechanics into geometrical theories of physics, new quantities must be included in the description of the infinitesimal structure of space-time. Using gauge transformations, Weyl's coefficients of connection are generalized to include not just the metric tensor and the electromagnetic potential, but the wave function as well. As the field equations are developed, it is found that invariance of the scalar of curvature under the appropriate gauge and coordinate transformations implies the Klein–Gordon equation. Since the terms required for invariance correspond with known quantum effects, no invariant classical limit is possible. The selection of appropriate fixed gauges and the elimination of small terms does lead directly to the Hamilton–Jacobi equations. Trajectories are defined for all cases, and as an example, the Lorentz force law is derived from these trajectories. The probability density is related to the particle trajectories and a conformal parameter of an additional metric tensor. Because of the gauge transformation, gravitational, electromagnetic, and quantum effects must be described as aspects of the same geometrical structure.

1. INTRODUCTION

The number of unified field theories advanced in the literature is large. The need is great as no known single theory predicts or explains everything observed. Because of the large number of attempts, one might conclude that such a theory is easy to write. On the contrary, it is no simple problem and many contesting points of view have remained since the issues were attacked earlier in this century. The only possible conciliation is further discussion, and yet, because the problem has remained incompletely solved for so long, an irrational faith is needed to believe that any solution at all can be found.

[1]Presently at General Electric Co., Edison Park, 8499 Darrow Road, Twinsburg, Ohio 44087.

An important step to resolve some of the conflicts that remain, is to put theories based on quantum mechanics and theories based on geometrical gravitation together in a consistent physically correct way. The manner to do this is uncertain since there are no glaring contradictory experimental data which might thrust inadequacies of the present theories into the open. Progress is needed but there are few clues to direct the improvements. There are no constraints to reduce and give form to the set of all possibilities. Some new assumption, right or wrong, is needed within which to discover new laws and relationships. It is for these reasons that the choice to use geometrical descriptions is made.

Certain opportunities are opened up by the geometrical assumption. First of all, the separation of physics into kinematics and dynamics may end. If the mass is allowed to become a parameter controlling the type and shape of the space in which the particles travel, then the motion itself, fixed by the shape of a coordinatelike structure can be considered both as a system of dynamics and as a system of kinematics. The difference is only in the way the interpretation is made.

Another favorable possibility is the separation of statistical effects and simple mechanical motion. In a geometrical theory, particles must have trajectories and the trajectories themselves must be assigned a distribution only after they are calculated. There is and must be an ultimate physical interpretation in terms of displacements and directed motion. The geometrical conceptualization is sufficiently straightforward that a statistical interpretation cannot be concealed.

A central difficulty is also created by the geometrical assumption. Quantum mechanics, as it has been developed and applied, has not been treated as a geometrical theory. One must either quantize gravity or geometrize quanta. The second option, selected here, might be a viable alternative.

The complexities of a complete unified theory are probably great, so only the simplest issues are addressed directly, no second quantization is used, and only one-particle equations are attempted. This single particle moves in, and is affected by three fields. The first of these is the gravitational field. For purposes of this article, all gravitational effects can be derived from a single macroscopic tensor $\dot{g}_{\mu\nu}$ which is the classical metric tensor. The electromagnetic field is also included because it is important for correctly deriving the form of the geometrical structure. It is introduced by way of the potential function ϕ_μ. The third field, the wave function ψ, completes the structure that is needed to define particle motion. ψ is a scalar for this discussion; spinor effects are ignored.

All of these fields are produced by other particles which are treated as simple classical point particles so as to provide well-defined external interactions.

2. WEYL'S THEORY EXTENDED[2]

Weyl's original theory uses concepts from non-Riemannian geometry (Eisenhart, 1972) to integrate electrodynamics into general relativity. Many of these concepts are useful here. The first of these is the conventional vector. Such a vector is generated from a small coordinate displacement. These vectors are the link to the physical ideas of magnitude and direction. The relationship between different basis vector systems at different points is defined and described by formal mathematical functions called connections. For nearby points, the connection $\Gamma^{\beta}_{\mu\nu}$ is the linear part of the effect of a displacement of a vector Δx^{μ} at a point P to a new point P',

$$\Delta x^{\mu'} \cong \Delta x^{\nu} \Gamma^{\mu}_{\nu\lambda}\, dx^{\lambda} + \Delta x^{\mu} \tag{1}$$

where the displacement vector is dx^{λ}. (The Einstein summation convention holds.) The two vectors $\Delta x^{\mu'}$ and Δx^{μ} are equal for dx^{λ} zero. Equation (1), to first order, defines the connections.

Various types of geometry are obtained by choosing different restrictions on the connections. The first important question is, given two vectors A and B at point P, will A displaced to the vertex of B point to the same place as B displaced to the vertex of A? A simple calculation following equation (1) shows that this will happen only if the connections are symmetric in the lower two indices. Since it is not necessary to use nonsymmetric connections for a description of the physical effects discussed, symmetry is assumed and one can make up a series of coordinate displacements and expect to arrive at the same final point irrespective of the order of execution. Other further constraints can be applied to the connections to restrict them even more. They are often formulated in terms of a third vector C which is displaced first by A and then by B. The result is compared to the vector obtained by the commuted displacements, B then A. If the results of the two displacements of C are of the same length, the space is Riemannian; if the results are equal (for all of the space and all pairs of displacements), the space is Euclidean. For Weylian geometry, the connections are assumed symmetric; the two displaced vectors might have different lengths and different directions. A more general theory involving an asymmetrical connection may be needed in a unified field theory but should be developed as the physical structure becomes apparent (Einstein, 1950, Appendix II).

[2]Weyl, (1952), Chap. 2; Dirac (1973).

If the connections are chosen according to the Riemannian prescription,

$$\dot{\Gamma}^{\lambda}_{\mu\nu} = \left\{ \begin{matrix} \dot{\lambda} \\ \mu\nu \end{matrix} \right\} \equiv \dot{g}^{\lambda\tau} \frac{1}{2} \left(\frac{\partial \dot{g}_{\tau\nu}}{\partial x^{\mu}} + \frac{\partial \dot{g}_{\mu\tau}}{\partial x^{\nu}} - \frac{\partial \dot{g}_{\mu\nu}}{\partial x^{\tau}} \right) \tag{2}$$

then the $\Gamma^{\lambda}_{\mu\nu}$ are intrinsically symmetrical but there is insufficient space for the electromagnetic potential.

With the structural change accompanying Weylian connections, the two-world concept of Weyl becomes relevant (Weyl, 1952, Sec. 35). One of these worlds consists of a "natural geometry," which corresponds to the intuitive, classical macroscopic world. A different world or metaphysical view on the same observations is the "world geometry," which corresponds to the detailed structure, inner workings, and physical–mathematical causes of the existence of the "natural geometry." Any sort of laws might be supposed for the microscopic "world geometry," as it is, by definition, inaccessible directly. Its justification lies in the success of any attempts to explain physical laws that might be formulated using it. Such an epistemological extension occurs often in physics, although it is not always explicit. None of the known fundamental fields are directly observable and all are created for the purposes of deducing more tangible effects. Weyl's "world geometry" or microscopic world is an appropriate place in which to define and claim existence for quantum trajectories. This microscopic world is geometrical and these trajectories, although not directly observable, can be conceptually useful.

Because of the two geometrical levels, a notational aid is used to mark quantities which refer more to one or the other of them. Quantities with dots (Adler et. al., 1965, Chap. 13) refer to the macroscopic world and follow, in that approximation, known macroscopic laws. Quantities without dots are part of the world geometry and, although they may obey known physical laws, they are part of this different structure. The separation is not completely unique nor are both versions of a quantity always used. Some symbols are useful for both geometries.

The contravariant coordinates x^{μ} are unique and common to both geometrical levels. The auxiliary Riemannian metric $\dot{g}_{\mu\nu}$ is very useful to define the way an observer would measure some macroscopic property of a trajectory, possibly either a curvature or a probability distribution. This tensor is not conceptually equivalent to the tensor $g_{\mu\nu}$, which involves a particular particle alone.

The microscopic quantities ψ, A_{μ}, $g_{\mu\nu}$ must be combined into coefficients of connection. The way to do this is suggested in Weyl's work.

Among the things that Weyl studied was the meaning (or absence of meaning) of the absolute scale of the metric tensor. He argued that for any arbitrary function $\lambda(x)$, the two metrics $g_{\mu\nu}(x)$ and $\lambda(x)g_{\mu\nu}(x)$ should be physically the same for a fixed coordinate space. There is no way to observe the absolute magnitude of $g_{\mu\nu}$ using conventional nonquantum measurements, only the relative direction of four-vectors can be converted into a number.

Weyl configured his connections so that functional multiplication of $g_{\mu\nu}(x)$ would be allowed mathematically but only if another vector, included in the connections, was changed simultaneously. This vector, ϕ_μ, has properties very much like the electromagnetic potential. The gradient part of this electromagnetic potential would change to compensate for changes in the "recalibration" of $g_{\mu\nu}$. In this way at least the two fields were combined into one unit, the connections; and any equation, expressed in terms of the connections or other proper combinations of the two fields, would be invariant under this process of recalibration or gauge transformation. Gauge, to Weyl, came from the gauging properties of the metric tensor. The connections, which are the fundamental quantities of geometry, would remain the same and would allow a gauge transformation for mathematical convenience.

To develop the connections for all three fields, some changes in physical interpretation are appropriate. The quantum field corresponds loosely to the overall gauge, which according to Weyl is unobservable. That is, if $g_{\mu\nu}$ were multiplied by a function $\lambda(x)$, and no compensating change were made in the other fields, the resulting change in the connections would cause physical effects equivalent to a change in the wave function. The absolute magnitude of $g_{\mu\nu}$ has, in combination with the other quantities in the connection, some physical significance. Such a change would have to be inferred, just as is done in quantum mechanics, from statistical measurements of probability densities.

As was shown by Weyl, to allow canceling changes in the fields $g_{\mu\nu}$ and ϕ_μ, the connections must have the form

$$\Gamma^{\beta}_{\nu\lambda} = \left\{ \begin{matrix} \beta \\ \nu\lambda \end{matrix} \right\} + \delta^{\beta}_{\mu}\phi_{\nu} + \delta^{\beta}_{\nu}\phi_{\mu} - g_{\mu\nu}\phi^{\beta} \tag{3}$$

The transformation properties of this quantity and the invariant tensors that can be generated from it correspond exactly with electromagnetic theory insofar as it is possible to tell. What is not worked out in Weyl's early papers is the exact relationship of ϕ_ν to the electromagnetic potentials.

An extension of the connections of equation (3) can be made by using the usual quantum transformation between the electromagnetic field and wave function. This brings the wave function into the structure quickly and in the simplest possible correct way:

$$\Gamma_{\mu\nu}^{\beta}=\left\{\begin{matrix}\beta\\ \mu\nu\end{matrix}\right\}+D_{\mu\nu}^{\beta} \tag{4}$$

for which

$$D_{\mu\nu}^{\beta}=\delta_{\mu}^{\beta}(\phi_{\nu}-\ln\psi|_{\nu})+\delta_{\nu}^{\beta}(\phi_{\mu}-\ln\psi|_{\mu})-g_{\mu\nu}(\phi^{\beta}-\ln\psi|^{\beta}) \tag{5}$$

This application of the minimal substitution anticipates the relationship of ϕ_{μ} to A_{μ} defined by equation (29). The vertical bar is defined in equation (11) and indices can be raised and lowered by the tensor $g_{\mu\nu}$. With this form for the connections, mathematical development can begin.

3. THE MATHEMATICS OF WEYL'S THEORY[3]

As in general relativity, arbitrary coordinate transformations are allowed which can be expressed in terms of the contravariant coordinates as

$$x^{\mu'}=f^{\mu}(x^{\nu}) \tag{6}$$

(Greek indices extend over the four dimensions of space-time.) Tensors, which form a class of important physical quantities, must, by definition, transform according to a specific law. A tensor of the general form $T_{\ldots}{}^{\mu}{}_{\ldots\nu\ldots}(x^{\beta})$ becomes $T'_{\ldots}{}^{\mu}{}_{\ldots\nu\ldots}(x^{\beta'})$ at the transformed point $x^{\beta'}$ and is given by

$$T'_{\ldots}{}^{\mu}{}_{\ldots\nu\ldots}(x^{\beta'})=T_{\ldots}{}^{\tau}{}_{\ldots\rho\ldots}(x^{\beta})\cdots\frac{\partial x'^{\mu}}{\partial x^{\tau}}\cdots\frac{\partial x^{\rho}}{\partial x'^{\nu}}\cdots \tag{7}$$

Tensors can be generated from other tensors by covariant differentiation. The coefficients of connection defined earlier will describe the behavior of the basis vectors when displaced by an infinitesimal amount and are needed to give the full effect of the differentiation.

The coefficients of connection are not tensors but transform according to a different law:

$$\Gamma_{\mu\nu}^{\beta'}=\Gamma_{\rho\sigma}^{\lambda}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x'^{\beta}}{\partial x^{\lambda}}-\frac{\partial^{2}x'^{\beta}}{\partial x^{\rho}\partial x^{\tau}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\nu}} \tag{8}$$

[3]Weyl (1952), Chap. 2.

This law is the same as for the Christoffel symbols $\{{}^{\beta}_{\mu\nu}\}$ hence the additional Weyl contribution to the coefficients of connection $D^{\beta}_{\mu\nu}$ is a tensor.

The covariant derivative of a vector field can be defined in terms of these quantities by.

$$V_\mu|_\nu = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\beta_{\mu\nu} V_\beta \tag{9}$$

The first term on the right represents the intrinsic change in the field while the second term represents the effect of the change in coordinate basis systems. The generalization of this differentiation to the multi-index tensors is analogous to Riemannian geometry except that the metric tensor itself has nonzero covariant derivative due to the terms in ϕ_μ and ψ in the connections. A direct application of (9) to each index of $g_{\mu\nu}$ gives (Eisenhart, 1972, Sec. 30)

$$g_{\mu\nu}|_\beta = 2g_{\mu\nu}(\phi_\beta - \ln\psi|_\beta) \tag{10}$$

Of course,

$$\ln\psi|_\beta = \frac{\partial}{\partial x^\beta} \ln\psi \tag{11}$$

The covariant derivative of a scalar field still equals the partial derivative. Weyl's connections affect the way the basis vectors change at proximate points and not the intrinsic variation of fields.

In Section 8 a velocity vector field is derived uniquely from a given set of coefficients of connection. Because the equation of motion comes from the connections alone, displacements of the actual particle for the possible purpose of measuring or observing the connections is not physically allowed. The extended connections only make physical sense if they are placed wholly within the microscopic "world." They can only be observed by implication. The motion of the particle is so strongly constrained by the connections that it is impossible to discuss physical changes in direction of the particle without a change of connections. In spite of the nonmaterial nature of the space, it is helpful to be able to describe and use a structure defined by the connections as if it were an object.

As mentioned earlier, a vector displaced around a small loop and compared to itself, need not agree in either length or direction with itself before displacement. The various changes placed upon the vector can be said to come from different parts of the connections. To see the effect of the fields, one simply displaces a vector along a curve and studies the character of the lowest-order effect. The mathematical behavior can be used to separate the three fields.

(1) The Gravitational Field. The tensor $g_{\mu\nu}$ has the same effect as the tensor $\dot{g}_{\mu\nu}$ of the Riemannian world. Turning off the quantum and electromagnetic field by setting $\phi_\mu = 0$, $\psi = 1$ leaves Weyl's connections equal to the Christoffel symbols.

(2) The Electromagnetic Field. The gravitational field and quantum field can be turned off by setting

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix} \tag{12}$$

and $\psi = 1$, respectively. The remaining "electromagnetic-Euclidean" connections cause only a relative change in length of a vector. Since distance is integrable in the external space, $\dot{g}_{\mu\nu}$ can be used to observe the effect of the displacement (Adler et al., 1965, p. 403):

$$\delta |V|^2 = \delta\left(V^\mu \dot{g}_{\mu\nu} V^\nu\right) = 2\dot{g}_{\mu\nu} V^\mu|_\beta V^\nu \delta x^\beta = 2\dot{g}_{\mu\nu} V^\mu V^\nu \phi_\beta d_x^\beta \tag{13}$$

for displacement δx^β. In simple notation, the length V of a vector changes by

$$\frac{\delta V}{V} = \phi_\beta x^\beta \tag{14}$$

ϕ^β measures the linear fractional coefficient of change in length. Since ϕ^β is presumably arbitrary, it is not an exact differential and the integrated effect may depend on the path that is taken to the displaced point. This is true even when observed with $\dot{g}_{\mu\nu}$. The property of nonintegrability is necessary to produce the effects of the magnetic field.

(3) The Wave Function. The logarithmic derivative $\ln\psi|_\mu$ has, except for sign, the same effect as the electromagnetic potential. It is always integrable, so that an expression of the form

$$\frac{\delta V}{V} = -\frac{\delta\psi}{\psi} \tag{15}$$

holds in the absence of an electromagnetic field [London, 1927, equation (8)]. The length of the displaced vector then has a unique size relative to the wave function. ψ remains a simple scalar field.

4. GAUGE TRANSFORMATIONS

Usually a gauge transformation is a change of the field variables depending on arbitrary functions which leave a physical law invariant. The best known of these is the classical electromagnetic gauge transformation for which the electromagnetic potential is changed according to

$$A'_\mu(x) = A_\mu(x) + B_\mu(x)$$

Of course, if the electromagnetic field and the associated forces are to be invariant, then the vector B_μ must be equal to the gradient of some scalar function. In a similar way, the gauge structure of this theory involves transformations of the fields which leave certain quantities invariant. A minimum requirement is that the connections should be unchanged. A gauge transformation affects neither the geometrical structure of the space nor the coordinates x^μ. In this way, physical quantities derived directly from the connections will have the correct gauge properties. There are two independent gauge transformations that are used implicitly in Section 2 to derive the connections. They can be parameterized by two functions of the coordinates, $f = f(x^\mu)$ and $h = h(x^\mu)$. The first is the quantum mechanical gauge transformation which is usually associated with the minimal substitution (Weyl, 1950b, p. 100)

$$\begin{aligned} \phi'_\mu &= \phi_\mu - \ln h|_\mu \\ \psi' &= h\psi \end{aligned} \tag{16}$$

The other gauge transformation, first discussed by Weyl (Weyl, 1950a, p. 207) is associated with his process of recalibration:

$$\begin{aligned} g''_{\mu\nu} &= f^2 g_{\mu\nu} \\ \phi''_\mu &= \phi_\mu + \ln f|_\mu \end{aligned} \tag{17}$$

These two can be combined formally to give a more general transformation:

$$\begin{aligned} g'_{\mu\nu} &= f^2 g_{\mu\nu} \\ \phi'_\mu &= \phi_\mu + \ln(f/h)|_\mu \\ \psi' &= h\psi \end{aligned} \tag{18}$$

Note that setting $f = h = d$ gives a transformation which does not involve the electromagnetic potential:

$$\bar{g}_{\mu\nu} = d^2 g_{\mu\nu}$$
$$\bar{\psi} = d\psi \tag{19}$$
$$\bar{\phi}_\mu = \phi_\mu$$

All of the transformations (16)–(19) are identical invariants of the connections (4). Two of these are not physically important. Since it is shown later that ϕ_μ, for physical reasons, should be pure imaginary, h in (16) should be a phase function of the form $e^{ib(x)}$. Since $g_{\mu\nu}$ should remain real, (17) and (18) should not be used in general. The inferred transformation (19) is allowed for real h; only (16) and (19) are usually used when a change of gauge is needed. The magnitude of the wave function transforms with the real metric tensor and the phase of the wave function transforms with the electromagnetic vector potential.

The additional transformations (17) or (18) are present as formal invariants. Precise limits of the gauge group can be defined by choosing an arbitrary connection $\Gamma^\beta_{\mu\nu}$ and ascertaining what possible fields can make it up. Let there be a connection given in terms of three fields $g_{\mu\nu}$, ϕ_μ, ψ and suppose that three other fields $g'_{\mu\nu}$, ϕ'_μ, $\psi'$are sought which might produce the same numerical values of $\Gamma^\beta_{\mu\nu}$. Firstly, gauge transformations (16) and (19) can be used to eliminate both wave functions in so far as they contribute to the connections. New fields $g^*_{\mu\nu}$, ϕ^*_μ and $g'^*_{\mu\nu}$, ϕ'^*_μ are generated. Since ϕ_μ is pure imaginary, the tensor $D^\lambda_{\mu\nu}$ is also pure imaginary. Dropping asterisks, the imaginary parts of the two tensors must be equal,

$$D^\lambda_{\mu\nu} = D'^\lambda_{\mu\nu} \tag{20}$$

as well as the real Christoffel symbols:

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}' \tag{21}$$

The equality of the two contracted terms $D^\lambda_{\mu\nu}$ and $D'^\lambda_{\mu\nu}$ shows the equality of ϕ_μ and ϕ'_μ.

The equality of the two metric tensors $g_{\mu\nu}$ and $g'_{\mu\nu}$ does not follow from the equality of the Christoffel symbols. The two tensors need only be related by an affine transformation. Such a transformation is equivalent to a subgroup of the group of all coordinate transformations. The total of gauge transformations which cannot be produced by equivalent coordinate transformations is given by (16) and (19).

Some of the simplicity of geometrical interpretation is lost when the vector potential and the connections become complex. Fortunately, the mathematics behaves like the original (all real) Weyl theory. This similarity is suggestive and useful. Because all of these quantities belong to the "world geometry" of Weyl, the complexification is not prohibited by any physical arguments. The explicit use of complex numbers in the coefficients of connection cannot be avoided because of the important part they play in quantum mechanics.

5. USEFUL TENSORS

The important physical tensors for single-particle motion can be constructed from the generalized Riemann curvature tensor. If a vector V^μ is displaced around an infinitesimal parallelogram $dx^\mu\, dx^\nu$ it will be changed relative to its starting value by an infinitesimal amount. This change when calculated to first order provides the definition of the curvature tensor. It is entirely analogous to the curvature tensor of general relativity:

$$\delta V^\beta = \left(V^\beta|_{\mu\nu} - V^\beta|_{\nu\mu} \right) dx^\nu\, dx^\mu \equiv R^\beta_{\tau\mu\nu} V^\tau\, dx^\mu\, dx^\nu \tag{22}$$

Because a linearly independent set of pairs $dx^\nu\, dx^\mu$ can be chosen,

$$R^\beta_{\tau\mu\nu} V^\tau = V^\beta|_{\mu\nu} - V^\beta|_{\nu\mu} \tag{23}$$

Since V^τ is an arbitrary vector field, the right-hand side of this can be expanded and the vector field eliminated:

$$R^\beta_{\mu\nu\rho} = \frac{\partial \Gamma^\beta_{\mu\nu}}{\partial x^\rho} - \frac{\partial \Gamma^\beta_{\mu\rho}}{\partial x^\nu} + \Gamma^\beta_{\tau\rho}\Gamma^\tau_{\nu\mu} - \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\rho\mu} \tag{24}$$

When written in this form, the gauge invariance of $R^\beta_{\mu\nu\rho}$ is certain since it depends only on $\Gamma^\beta_{\mu\rho}$.

The physically important contraction with respect to the first lower index relates to the change in length of a vector for a small displacement. It can be calculated easily from (24):

$$R^\mu_{\mu\nu\rho} = \frac{\partial \phi_\rho}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\rho} = H_{\nu\rho} \tag{25}$$

The symmetry properties of the $H_{\rho\nu}$ tensor are correct for an electromagnetic field. It is imaginary when measured in geometrical units because ϕ_μ is imaginary.

The contraction of the Riemann tensor between the first and third indices or first and fourth indices gives a generalized Ricci tensor,

$$R^{\beta}_{\mu\nu\rho} = Q^{\beta}_{\mu\nu\rho} - D^{\beta}_{\mu\nu}|_{\rho} + D^{\beta}_{\mu\rho}|_{\nu} - D^{\beta}_{\rho\tau} D^{\tau}_{\mu\nu} + D^{\beta}_{\tau\nu} D^{\tau}_{\rho\mu} \tag{26}$$

where $Q^{\beta}_{\mu\nu\rho}$ is the Riemann tensor formed from the Christoffel symbols of the tensor $g_{\mu\nu}$. The contraction can be found in covariant form:

$$R_{\mu\rho} = R^{\nu}_{\mu\nu\rho} = Q_{\mu\rho} - 2H_{\mu\rho} - \tilde{\phi}_{\mu}|_{\rho} - \tilde{\phi}_{\rho}|_{\mu} - g_{\mu\rho}\tilde{\phi}_{\nu}|^{\nu} + 2\tilde{\phi}_{\mu}\tilde{\phi}_{\rho} - 2g_{\mu\rho}\tilde{\phi}_{\nu}\tilde{\phi}^{\nu} \tag{27}$$

where $Q_{\mu\rho} = Q^{\nu}_{\mu\nu\rho}$ and $\tilde{\phi} = \phi_{\mu} - \ln\psi|_{\mu}$. $R_{\mu\nu}$ is Hermitian. Note that with this definition of covariant derivative, the tensor $g_{\mu\nu}$ does not commute with the vertical bar. Equation (10) must be used.

6. FIELD EQUATIONS FOR $g_{\mu\nu}$, ϕ_{μ}, AND ψ

Einstein's gravitational field equations may be used for the purpose of this article to supply the macroscopic tensor $\dot{g}_{\mu\nu}$. Avoiding the complexity associated with this calculation, we assume that it is a known, given quantity. Measurement of this tensor involves many particles, either as components of the clocks which measure the time or as absorbers and emitters of the measuring light beams. A measurement of $\dot{g}_{\mu\nu}$ thereby always involves a number of quantum states and is not a fundamental one-particle quantity.

Without a complete microscopic gravitational theory, an assumption involving multiple quantum states is needed to define the one-particle connection from $\dot{g}_{\mu\nu}$. It should be possible with a more comprehensive theory to justify the use of $\dot{g}_{\mu\nu}$ by considering collections of states. In this approximation, $\dot{g}_{\mu\nu}$ must introduce the gravitational effects and, if these are to be the correct gravitational effects, the tensor $g_{\mu\nu}$ must be related to the tensor $\dot{g}_{\mu\nu}$. Certainly, if $g_{\mu\nu} = \dot{g}_{\mu\nu}$, the correct force law will be obtained; however, in light of the gauge transformation (19), any assumption $g_{\mu\nu} = \xi^2 \dot{g}_{\mu\nu}$, in which ξ is an arbitrary function of coordinates, will restrict the possibilities for the functions $\Gamma^{\beta}_{\mu\nu}$ in exactly the same way. It is often convenient to let $g_{\mu\nu} = \dot{g}_{\mu\nu}$, but the more general form is more useful for the interpretation of Section 9. More importantly, if the gauge in which $\psi = 1$ is used, the ξ factor must be allowed.

The electromagnetic vector potential can be specified by using the contracted Riemann–Weyl tensor $H_{\tau\rho}$. Maxwell's equations can be adapted to this tensor and are given by

$$\frac{\partial}{\partial x^{\nu}}\left[H_{\sigma\rho}\dot{g}^{\sigma\mu}\dot{g}^{\rho\nu}(-\dot{g})^{1/2}\right] = 4\pi i\alpha j^{\mu} \tag{28}$$

The raising of indices is done in the Riemann space of the observer. The units are cgs and have been chosen so that j^μ is the particle current density as measured for particles of electromagnetic interaction $\alpha = e^2$. Since j^μ is real, ϕ_μ can be selected pure imaginary so that it is related to the usual vector potential A_μ by

$$A_\mu = \phi_\mu / ie \tag{29a}$$

These units are chosen so that equation (35) below will appear in normal form. The geometrical tensor $H_{\mu\nu}$ is related to the physical tensor $F_{\mu\nu}$ by the same factor:

$$F_{\mu\nu} = H_{\mu\nu} / ie \tag{29b}$$

The usual equation of the wave function in quantum mechanics is made in a fixed gauge much as an actual calculation of the action function is made in a fixed gauge. The larger gauge group of this theory can be kept if it is understood that additional constraints on the fields are needed to specify a particular gauge.

Suppose then that particular solutions for A_μ and $g_{\mu\nu}$ are chosen. Since ψ is a complex scalar, one complex equation will suffice for its calculation. Weyl has suggested that the generalized scalar of curvature given by

$$R = R_{\mu\nu} g^{\mu\nu} \tag{30}$$

could be set equal to a constant (Eddington, 1975, Sec. 89). In his discussion, no further interpretation is given except to say that this sets some absolute gauge or standard of length for space-time (Weyl, 1923, p. 215). To Weyl, the absolute gauge was not physically important; however, because of the relation of $|\psi|$ to the size of $g_{\mu\nu}$, the absolute gauge (of Weyl) should be important for quantum mechanics. An equation based on this curvature scalar might be appropriate. Weyl realized a problem, which is that $R_{\mu\nu} g^{\mu\nu}$ is not gauge invariant (Weyl, 1952, p. 134). The interpretation is in difficulty if the gauge transformations (16) and (19) are to be physical invariants. If, however, the apparent curvature scalar, as measured from the external Riemannian space, $R_{\mu\nu} \dot{g}^{\mu\nu}$, is set equal to a constant, these formal difficulties are avoided. Setting this equal to a constant results in a gauge invariant constraint on the space as defined by the coefficients of connection. In a fixed gauge, this constraint can be expressed as a differential equation for the wave function. Let then the gauge be restricted by $g_{\mu\nu} = \dot{g}_{\mu\nu}$ and suppose that the scalar formed by contraction of the Ricci tensor is the constant $-6m^2$ in which m is the mass of the particle with wave function ψ. The resultant equation is

$$-6m^2=R_{\mu\nu}\dot{g}^{\mu\nu}=R_{\mu\nu}g^{\mu\nu}=\dot{R}-\frac{6}{(-g)^{1/2}}\frac{\partial}{\partial x^{\rho}}\left[\sqrt{(-g)}^{1/2}\tilde{\phi}^{\rho}\right]+6\tilde{\phi}^{\rho}\tilde{\phi}_{\rho} \tag{31}$$

in which $g=\det(g_{\mu\nu})$, $\dot{R}$ is the Riemannian curvature formed with respect to the metric $\dot{g}_{\mu\nu}$, and once again $\tilde{\phi}_{\rho}=\phi_{\rho}-\ln\psi|_{\rho}$. Substituting for ϕ_{μ} and rearranging gives

$$\begin{aligned}-m^2=&\left(\phi_{\mu}-\frac{1}{\psi}\frac{\partial\psi}{\partial x^{\mu}}\right)\left(\phi^{\mu}-\frac{1}{\psi}\frac{\partial\psi}{\partial x_{\mu}}\right)\\&-\frac{1}{(-g)^{1/2}}\frac{\partial}{\partial x^{\mu}}\left[(-g)^{1/2}\left(\phi^{\mu}-\frac{1}{\psi}\frac{\partial\psi}{\partial x_{\mu}}\right)\right]+\frac{\dot{R}}{6}\end{aligned} \tag{32}$$

which in operator notation is

$$-\left(m^2+\frac{\dot{R}}{6}\right)\psi=\frac{1}{(-g)^{1/2}}\left(\frac{\partial}{\partial x^{\mu}}-\phi_{\mu}\right)(-g)^{1/2}\left(\frac{\partial}{\partial x_{\mu}}-\phi^{\mu}\right)\psi \tag{33}$$

Restoring units according to (19) so that

$$\frac{\partial}{\partial x^{\nu}}\left[F_{\sigma\rho}\dot{g}^{\sigma\mu}\dot{g}^{\rho\nu}(-\dot{g})^{1/2}\right]=4\pi e j^{\mu} \tag{34}$$

makes equation (33) into

$$\left(m^2+\frac{\dot{R}}{6}\right)\psi=\frac{1}{(-g)^{1/2}}\left(\frac{1}{i}\frac{\partial}{\partial x^{\mu}}-eA_{\mu}\right)(-g)^{1/2}\left(\frac{1}{i}\frac{\partial}{\partial x_{\mu}}-eA^{\mu}\right)\psi \tag{35}$$

This is the Klein–Gordon equation (Bjorken and Drell, 1964, pp. 5–6) including the minimal substitution for the electromagnetic potential except for the small term $\dot{R}/6$ (Penrose, 1964). If $\dot{R}$ is macroscopic in size, say about 1 cm^{-2}, the perturbation to the mass is $\Delta m/m=\dot{R}/12\, m^2)\sim 10^{-22}$, which is well beyond any experimental tests even with this conservatively high value for $\dot{R}$. This leaves $m^2\psi$ as the dominant term on the left side of equation (35). The mathematics of quantum mechanics is equivalent to the study of spaces of constant scalar curvature.

It is indeed interesting that out of a complex of nonlinear equations, there should be found an exact linear equation such as (35). The nonlinearities are all contained in the way in which the electromagnetic and gravitational terms enter into the equation. In this sense quantum mechanics is simply the largest piece of Weyl's theory that can be represented by a linear field equation. Even though it combines the physically very dissimilar phase and amplitude, it is not especially surprising that it should be discovered before difficult nonlinear equations that include gravitation and electrodynamics.

Solutions of this equation are now well studied and the interpretations are for the most part dependent on the equation itself. Many applications depend on known interactions of the electromagnetic field with the particles. Equation (35) can not be verified easily without recourse to this interaction.

The derivation of equation (35) is the basic result of this article. The remaining sections concern extensions of this interpretation to give a consistent geometrical explanation.

7. THE CLASSICAL FIELD EQUATIONS

Usually the classical limit is found by eliminating terms proportional to $\hbar$. In particular, the last two terms of equation (32) are proportional to $\hbar$ and $\hbar^2$, respectively, when the variables are expressed in macroscopic units. These terms are negligible in the classical limit; unfortunately, they are necessary for the invariance of equation (32) under the coordinate transformation (6) as well as the gauge transformation (16) through (19). As a result, it is not possible to make the classical correspondence by discarding these terms and yet preserve the geometrical invariances of the theory. There is no gauge and coordinate transformation invariant classical limit for this Weyl-type theory. The classical limit must be described as an idealization of a mathematical structure which is in fact irreplaceably quantum mechanical.

The proper way to see the classical limit is to accept the fact that $\hbar$ is a nonzero constant and make appropriate physical assumptions to justify that the solution of equations (28) and (32) are close to another set of equations, namely, the equations of Jacobi (Landau and Lifshitz, 1962, p. 68). It is not possible to assume nor necessary to prove that the field variables are in fact solutions of the classical equations. They are only approximate solutions.

Assuming in a fixed region of space that the variations in the field are sufficiently slow so that the last two terms in equation (21) are negligible and supposing that the gauges are selected so that A_μ is real and $g_{\mu\nu}=\dot{g}_{\mu\nu}$, then making the substitutions $\psi=e^{is}$ and $\phi_\mu=ieA_\mu$ gives

$$\left(\frac{\partial S}{\partial x^\mu}-eA_\mu\right)\left(\frac{\partial S}{\partial x_\mu}-eA^\mu\right)=m^2 \tag{36}$$

The elimination of higher-order terms in equation (21) has also eliminated all explicit reference to complex numbers. In the classical limit, S is real up to an arbitrary additive complex constant.

Having derived this field equation, a complete description of a particle requires an equation of motion; fortunately, there is a natural vector contained in the connections which gives the correct velocity field in the classical limit.

8. THE LORENTZ FORCE LAW

The concept of trajectories for both quantum mechanical motion and classical motion is discussed in detail in an earlier paper (Galehouse, 1981). Once the wave function or action function has been chosen, the equations of motion of the particle having that wave function are first order in analogy with Hamilton–Jacobi theory. The velocity vector usually chosen is the quantity $\partial S/\partial x^\mu - eA_\mu$ at least for the classical case. A natural gauge invariant quantity must be found which will equal this at least under circumstances when the classical approximation holds. Such a quantity is the vector formed by contraction from the connections (Dirac, 1937, 1951, 1952; London, 1961, Sec. 10)

$$V_\mu = \operatorname{Im}\Gamma^\beta_{\beta\mu} = \operatorname{Im}\left(\frac{1}{4}\left\{\begin{matrix}\beta\\ \beta\mu\end{matrix}\right\} - \phi_\mu + \ln\phi|_\mu\right) \tag{37}$$

The imaginary part of the vector has been taken to keep the trajectories real, a process which corresponds to the reseparation of the phase and amplitude information which is generated by solving the field equations.

The velocity vector (37) can, in fact, be simplified, especially for classical motion. In the gauge $g_{\mu\nu} = \dot{g}_{\mu\nu}$ the Christoffel symbol is pure real and almost always very small. When the wave function can be written in terms of a real action $e^{iS} = \psi$, the velocity reduces to the expected:

$$V_\mu = \frac{\partial S}{\partial X^\mu} - eA_\mu \tag{38}$$

Using the gauge transformation (16) written in terms of A_μ and S to eliminate S from equation (36) gives

$$A_\mu \dot{g}^{\mu\nu} A_\nu = m^2/e^2 \tag{39}$$

These gauge transformations work for field and motion equations alike and thereby S is also eliminated from (38). A normalized velocity can be defined by

$$U_\mu = V_\mu \left(V_\beta V_\delta g^{\beta\delta} \right)^{-1/2} \tag{40}$$

and with the help of (39) reduces (38) to the simple formula (London, 1961)

$$\frac{dx^\mu}{ds} = \frac{e}{m} A^\mu \tag{41}$$

A_μ is part of a non-Riemannian space. In general, the particle still has an apparent acceleration because A_μ is nearly arbitrary in the classical limit. This acceleration, due to the space itself, is usually explained as the effect of the Lorentz force. The particle does not travel on a geodesic as would be determined by the metric of the observer but has an acceleration given by the absolute derivative of equation (41) (Synge, 1971, p. 4):

$$\frac{d^2 x_\mu}{ds^2} = \frac{e}{m} A_\mu \bigg|_\beta \cdot \frac{dx^\beta}{ds} = \frac{e}{m} \left(\frac{\partial A_\mu}{\partial x^\beta} - \dot{\Gamma}^\nu_{\mu\beta} A_\nu \right) \frac{dx^\beta}{ds} \tag{42}$$

Explicit reference to $\dot{\Gamma}^\beta_{\mu\nu}$ can be eliminated because A^μ satisfies the field equation (39). Taking the absolute derivative of this equation along the path of the particle, we obtain

$$0 = 2 A_\nu |_\beta A^\nu = A^\beta \left(\frac{\partial A_\mu}{\partial x^\beta} - \dot{\Gamma}^\rho_{\mu\beta} A_\rho \right) \tag{43}$$

or, using the equation (41),

$$\frac{dx^\beta}{ds} \dot{\Gamma}^\rho_{\mu\beta} A_\rho = \frac{\partial A_\beta}{\partial x^\mu} \cdot \frac{dx^\beta}{ds} \tag{44}$$

Substituting (44) into (42) gives

$$\frac{d^2 x_\mu}{ds^2} = \frac{e}{m} \left(\frac{\partial A_\mu}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\mu} \right) \frac{dx^\beta}{ds} = \frac{e}{m} F_{\mu\beta} \frac{dx^\beta}{ds} \tag{45}$$

This is the Lorentz force law and it is gauge invariant as written. It is not correct when quantum effects are important since these terms have been dropped during the derivation of equations (36) and (38). In spite of the problems of invariance, a Weyl theory can give the correct classical force law when the right approximations are used.

It is appropriate to review why the nonintegrability of length cannot be observed by direct experiment. Once a set of connections is chosen, the particle trajectory through a given starting point is unique. It is never possible for a closed loop to be formed from legal trajectories—and a real particle can follow no others. The interpretation of these equations is deep; they are not defined in terms of the motion, they define the motion.

9. INTERPRETATION OF THE PROBABILITY DENSITY AS A SUM OF TRAJECTORIES

The concept of probability density can be expanded because the particles have trajectories. (See contrasting viewpoints given by Parks, 1964; Einstein, 1932; Einstein et al., 1935; Gottfried, 1966, Sec. 2; Feynman, 1965; etc.). The probability current vector

$$J_\mu = \frac{i}{2m}\left[\psi^*\left(i\frac{\partial}{\partial x_\mu} - eA^\mu\right)\psi - \psi\left(i\frac{\partial}{\partial x_\mu} + eA^\mu\right)\psi^*\right] \tag{46}$$

except for normalization, is equal, in the gauge where $g_{\mu\nu} = \dot{g}_{\mu\nu}$ to the vector field defined by equation (37). Taking the imaginary part in (37) corresponds to the conjugation and subtraction used in (46). The multiplicative factor used is required by the field equation and sets the density so that the current is conserved.

To reproduce this known quantum law from the trajectories, some initial distribution must be assumed. In the infinitesimal limit, a small uniform sphere of particles will move along a trajectory and change its projected area as the lines of motion become more and less dense. Each of the particles in this sphere must be considered a member of an ensemble element, the total of these elements belonging to one wave function. Any arbitrary initial density will change along a trajectory so as to be exactly proportional to the density calculated by (46) simply because (46) is conserved. The constant of proportionality relating the initial density to the calculated value will remain the same. To explain the relative normalization of probability density at spacelike separated points requires some recourse to physical argument. Global probability or interference requires coherence and that implies that the ensemble particles travel from a virtual point source. Such a source, whether accessible in practice or not, provides a unique place to relate the various local densities to each other. Equation (46) is therefore the global completion of the effect of equation (37). It can be derived from (37) by requiring a multiplicative factor which makes the current density vector a conserved current density over the region in which

ψ is defined. This globally defined probability density can be given a geometrical interpretation as a variation in the size of the microscopic metric tensor $g_{\mu\nu}$.

A simple example will illustrate this physical way to interpret a gauge transformation. In Figure 1, the source A emits particles which are diffracted by the small hole in the screen at B and then detected on the screen C. Suppose that the source is small enough that the collection of wave functions which are needed to predict the diffraction pattern are sufficiently similar that the pattern is not seriously washed out. Further suppose that the hole is small so that diffraction is significant and is macroscopically observable on the screen. This will certainly happen if the hole is as small as the wavelength. Suppose also that the screen is placed far enough from the hole that the diffraction effects are no longer important near the screen and the particles can be treated in the classical limit. Finally, suppose, at least near the screen, that electromagnetic effects are not present and that the distribution of particles depends only on the two fields ψ and $g_{\mu\nu}$.

Beginning in the fixed gauge $g_{\mu\nu}=\dot{g}_{\mu\nu}$, let equation (35) be solved for this arrangement so that the wave function is a known function of position. In a macroscopic neighborhood of a point $\dot{x}^\mu$ on the screen, the solution may be approximated by the Taylor expansion:

$$S(x^\mu)=\theta-i\sigma+p_\mu(x^\mu-\dot{x}^\mu) \tag{47}$$

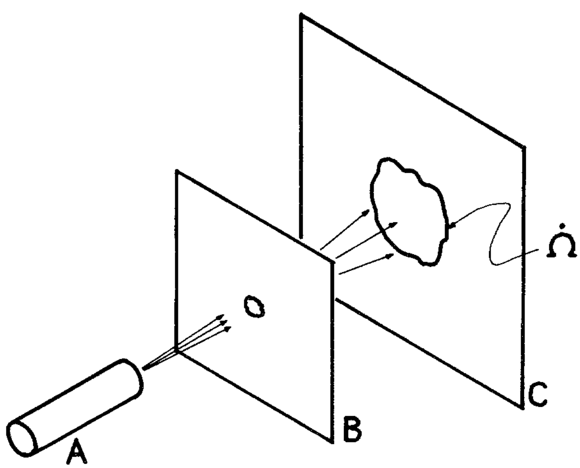

Fig. 1. A simple diffraction experiment consists of a source A, diffraction aperature B, and detection screen C. The probability density is measured by counting the particles which arrive at the screen within the region Ω.

θ and σ are real constants and p_μ is a real vector. The constants θ and σ certainly cannot be found by Hamilton–Jacobi theory but are known from the wave function since $\psi = e^{iS}$.

In the nonrelativistic limit, the time dependence is nearly e^{-imt} and the probability density, J_0, from (46) and (47) becomes

$$J_0 = \psi\psi^* = e^{-2\sigma} \tag{48}$$

and depends only on the zero-order coefficient σ. The probability density is not gauge invariant when expressed by equation (46). The gauge transformation (19) with $d = e^{-\sigma}$ may be applied to give a new ψ' and g' with $\psi\psi'^* = 1$ everywhere on the screen. The amplitude part of the wave function is transferred to the tensor $g_{\mu\nu}$:

$$g'_{\mu\nu} = e^{-2\sigma} g_{\mu\nu} \qquad \psi' = \exp\left[i(\phi + p\mu(x^\mu - \dot{x}^\mu))\right] \tag{49}$$

In this gauge, all variations in the particle density are due to variations in the metric tensor $g'_{\mu\nu}$. The parameter σ is called the conformal parameter in differential geometry (Eisenhart, 1926). A conformal change is equivalent to an isotropic expansion of the coordinate system. Since $g_{\mu\nu}x^\mu x^\nu$ is the square of a length, the effective normalization factor for a length at some point $\dot{x}^\mu$ is $e^{-\sigma}$. This rescaling of the local coordinate system does not change the effective velocity of the particle as

$$\frac{dx}{dt} = \frac{d(xe^{-\sigma})}{d(te^{-\sigma})} = \frac{d\bar{x}}{d\bar{t}} \tag{50}$$

The effective cross-sectional area $d\Omega$ does scale as $e^{-2\sigma}$ thereby giving the correct dependence on the conformal factor. In this guage the probability density variations come from equivalent variations in the metric tensor $g_{\mu\nu}$.

The probability density can be interpreted as a scale change or conformal change in the space in which the particle is moving. The lines of motion are compressed or rarified by contractions or expansions in the metric tensor.

The probability density is a direct geometrical concept depending on the motion of particles along trajectories. Quantum mechanics produces results very much like having an individual metric tensor for each particle of the form $|\psi|^2 g_{\mu\nu}$.

These two parts of the quantum mechanical wave function, the phase and the amplitude, are related to each other only through the field equation (35). Particle motion would be always classical if changes in velocity did not imply other variations in particle direction and density.

The probability density is the natural inherent density, characteristic of the geometrical structure underlying the space in which the particles are traveling.

10. CONCLUSION

One of the objects of this article is to develop certain epistemological problems of fields and particles. The separation between a quantum field and a classical field has been modified and simplified. A first-quantized particle has become unessentially different from an unquantized classical particle. And correspondingly, the trajectory of a first-quantized particle can be defined reasonably and with a result that is not surprising relative to a classical trajectory. This epistemology is augmented by the addition of the possibility that field and motion equations might in whole or part be of geometrical origin, or more simply, completely reducible to a geometrical description. Such a description is presented here in part for quantum, gravitational, and electromagnetic interactions. The close relationship of the trajectories with the quantum field equation suggests that the source equations of the electromagnetic and gravitational fields may be of simple geometrical origin as well. Certain proposals have been made, especially for the electromagnetic field, viz., Einstein (1950), Appendix II. The straightforward question is, whether any of these geometrical theories, each containing a correct element of the description of nature, can be put together in a physically sensible way. Insofar as this can be done, a number of developments may be possible. As seen already, the mass has become a part of the equations and not an *intrinsic* property of the particle. If physics is truly geometric, then all experiments can be defined by trajectories alone and the mass and interaction constants are part of the natural constraints on those trajectories. This suggests an end to separation of kinematics and dynamics and it also suggests that mass ratios and interaction constants might have geometrical origins. A physically correct inclusion of proper source equations with comparable motion equations would be at least required. Present-day quantum theories do not even provide the possibility of pursuing these questions theoretically.

Certain specific ideas from the philosophy of quantum mechanics and the philosophy of general relativity may need modification. A continuing problem for theories of combined relativity and quantum mechanics has been the conflict between the gravitational idea of a particle geodesic and the quantum idea of an operator. These basic philosophies are different as exemplified by the conflicting concepts of measurement. Many previous attempts have proceeded to impose one structure on the other. The essential conclusion here is that it is possible to synthesize these two theories and to put them together integrally without imposing one philosophy upon the other. A number of deep connections between the two exist, and, if these connections are relevant, any unified field theory that does not recognize them may likely fail. For this synthesis, certain constraints must be placed on the meaning of a general relativistic displacement and on the concept of a metric for a quantum particle; but, these are part of the macroscopic nature of general relativity and the microscopic nature of the quantum theory.

A number of unresolved issues from Weyl's work have been addressed. A specific value for the electromagnetic constants has been defined as opposed to simple arguments by invariance. The difference between natural geometry and world geometry and the source of that difference as quantum mechanics has been made more clear. Additionally, the probabilistic nature of quantum mechanics is separated from a deterministic equation of motion. An important addition to Weyl's theories is the equation of motion which allows the correct prediction of motion of charged particles including the otherwise anomalous effects of the magnetic field.

A number of difficulties remain which can now be formulated. (1) One can look for an explanation of the origin of the electromagnetic field source equation. This might be by way of a Weyl-type theory or perhaps with a structure similar to an asymmetrical connection. Very few past attempts include quantum effects in the simple way they are described here. It is possible that the inclusion of quantum effects in a simple way might improve the cogency and relevance of electromagnetic–gravitational theories. (2) As mentioned earlier, combined geometrical equations of the field and the motion may lead to predictions for the fundamental constants. (3) The question of spin has not been attacked. The author believes that the proper course is to ask how to integrate spin, quantum mechanics, general relativity, and electrodynamics into a single geometrical structure without imposing one set of ideas upon the others. (4) The problem of particle creation has been avoided because it always seems to lead to the necessity of handling infinite numbers of particles. Can this be simplified because the motion has been separated from the statistics?

This list is not exhaustive and these problems are exceedingly deep and solutions do not come quickly. A belief in the existence of solutions, even if irrational, is needed if answers are to be found. Certainly discussion of these fundamental questions has not been concluded.

ACKNOWLEDGMENTS

I wish to thank J. Krolik, Professor R. Sacks, and unknown reviewers for many helpful discussions as well as for making critical reviews of what would have been final manuscripts.

REFERENCES

Adler, R., Bazin, M., and Schiffer, M. (1965). *Introduction to General Relativity*. McGraw Hill, New York.

Bjorken, J. D., and Drell, S. D. (1964). *Relativistic Quantum Mechanics*. McGraw Hill, New York.

Dirac, P. A. M. (1937). *Nature*, **139**, 323.

Dirac, P. A. M. (1951a). *Nature*, **168**, 906.

Dirac, P. A. M. (1951b). *Proceedings of the Royal Society of London*, **209**, 291.

Dirac, P. A. M. (1952a). *Nature*, **169**, 146.

Dirac, P. A. M. (1952b). *Proceedings of the Royal Society of London*, **212**, 330.

Dirac, P. A. M. (1973). *Proceedings of the Royal Society of London*, **333** (1595), 403.

Eddington, A. S. (1975). *The Mathematical Theory of Relativity*. Chelsea, New York.

Einstein, A. (1960). At the 1932 Solvay Conference as recounted in L. DeBroglie, *Non-Linear Wave Mechanics*. Elsevier, New York, p. 74.

Einstein, A., Rosen, N., and Podolsky, B. (1935). *Physical Review*, **47**, 717.

Einstein, A. (1950). *The Meaning of Relativity*. Princeton University Press, Princeton, New Jersey.

Eisenhart, L. P. (1926). *Riemannian Geometry*. Princeton University Press, Princeton, New Jersey.

Eisenhart, L. P. (1972). *Non-Riemannian Geometry. American Mathematical Society*, Providence, Rhode Island.

Feynman, R. P., and Hibbs, A. R. (1965). *Quantum Mechanics and Path Integrals*. McGraw Hill, New York.

Galehouse, D. C. (1981). "Space-time trajectories of Quantum Particles," *International Journal of Theoretical Physics*.

Gottfried, K. (1966). *Quantum Mechanics Volume I: Fundamentals*. W. A. Benjamin, New York.

Landau, L. D., and Lifshitz, E. M. (1962). *The Classical Theory of Fields*. Addison Wesley, Reading, Massachusetts.

London, F. (1927). *Zeitschrift für Physik*, **42**, 375–389.

London, F. (1961). *Superfluids*, Volume 1, Dover, New York.

Parks, D. (1964). *Introduction to the Quantum Theory*. McGraw-Hill, New York.

Penrose, R. (1964). "Conformal Treatment of Infinity," in C. DeWitt, ed., *Relativity, Groups and Topology*. Gordon and Breach, New York.

Synge, J. L. (1971). *Relativity: The General Theory*. North Holland, Amsterdam.

Weyl, H. (1950a). In H. A. Lorentz, *The Principle of Relativity*. Dover, New York.

Weyl, H. (1950b). *The Theory of Groups and Quantum Mechanics*. Dover, New York.

Weyl, H. (1952). *Space, Time, Matter*. Dover, New York.